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Mh4714 Week 6

Week 6

0.1 Continuity

A function f is continuous at $a \in \mathbb{R}$ if:

(i) f is defined at a.

(ii) $\lim_{x \to a} f(x)$ exists.

(iii) $\lim_{x \to a} f(x) = f(a)$.

Example 0.1

(i) $\dfrac{\sin(x)}{x}$ is not continous at $x = 0$ because it is not defined there.

(ii)

$$f(x) = \begin{cases} \dfrac{1}{x} \text{ if } x \neq 0, \\ 1 \text{ if } x = 0. \end{cases}$$

is not continuous at $x = 0$ because $\lim_{x \to 0} \dfrac{1}{x}$ does not exist.

(iii)

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x} \text{ if } x \neq 0, \\ 2 \text{ if } x = 0. \end{cases}$$

is not continuous at $x = 0$ because $\lim_{x \to 0} \dfrac{\sin(x)}{x} \neq f(0)$

From the properties of limits listed above we can conclude the following about continuous functions:
Let f and g be continuous at $a \in \mathbb{R}$:

1. $f+g$ is continuous at a.
2. fg is continuous at a.
3. $\dfrac{f}{g}$ is continuous at a if $g(a) \neq 0$.
4. If the composition $f \circ g$ is defined over an open interval containing $a \in \mathbb{R}$ then $f \circ g$ is continuous at a if g is continous at a and f is continous at $g(a)$.

Example 0.2

$f(x) = x$ is continuous at every $a \in \mathbb{R}$ because

$$\lim_{x\to a} f(x) = \lim_{x\to a} x = a = f(a).$$

$g(x) = k$ ($k \in \mathbb{R}$) is continuous at every $a \in \mathbb{R}$ because

$$\lim_{x\to a} g(x) = \lim_{x\to a} k = k = g(a).$$

It follows that every polynomial is continuous over $\mathbb{R}$ because a polynomial is made up of sums of products of constants and the function x.

Example 0.3

Since every polynomial is continuous everywhere it follows from poperty (iii) above that a quotient $\dfrac{P(x)}{Q(x)}$ of polynomials $P(x), Q(x)$ is also continuous at every $x \in \mathbb{R}$ for which $Q(x) \neq 0$.

It can also be shown that the trigonometric functions $\cos(x), \sin(x)$ are continuous at every $x \in \mathbb{R}$.

0.1.0.1 Continuity over an interval.

Definition 0.4

If f is continuous at each element of the open interval (a, b) then f is said to be *continuous over* (a, b).

Definition 0.5

If f is continuous at each element of the open interval (a, b) and if $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$ then f is said to be *continuous over* $[a, b]$.

0.1.1 Intermediate Value Theorem

Theorem 0.6 (Intermediate Value Theorem)

Let f be a real valued function continuous over the interval $[a, b]$ with $f(a) \neq f(b)$ and let k be any number between $f(x)$ and $f(b)$.There is $c \in [a, b]$ with $f(c) = k$.

This theorem basically says that a continuous function takes on all values intermediate to its end-point values.

It is an important fact that the proof of this theorem depends on the *Completeness Axiom.*

Example 0.7

The function $f(x) = x^2,\ x \in \mathbb{Q}$ does not have this intermediate value property. For instance, $f(1) = 1, f(2) = 4$, $1 < 2 < 4$ but there is not c in the domain of f such that $f(c) = 2$. That is, f does not take on the value 2 even though 2 is between $f(1)$ and $f(2)$.